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# Symmetries of the stationary Einstein–Maxwell–dilaton Theory

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## Abstract

Gravity coupled three-dimensional  $\sigma$ -model describing the stationary Einstein–Maxwell–dilaton system with general dilaton coupling is studied. Killing equations for the corresponding five-dimensional target space are integrated. It is shown that for general coupling constant  $\alpha$  the symmetry algebra is isomorphic to the maximal solvable subalgebra of  $sl(3, R)$ . For two critical values  $\alpha = 0$  and  $\alpha = \sqrt{3}$ , Killing algebra enlarges to the full  $sl(3, R)$  and  $su(2, 1) \times R$  algebras respectively, which correspond to five-dimensional Kaluza–Klein and four-dimensional Brans–Dicke–Maxwell theories. These two models are analyzed in terms of the unique real variables. Relation to the description in terms of complex Ernst potentials is discussed. Non-trivial discrete maps between different subspaces of the target space are found and used to generate new arbitrary- $\alpha$  solutions to dilaton gravity.

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# 1 Introduction

Hidden symmetries in gravity and supergravity theories dimensionally reduced to three and two dimensions were extensively studied earlier [1], [2], [3], [4], [5], [6]. As it is well-known, vacuum Einstein equations in a space-time of arbitrary dimensions possessing a sufficient number of Killing vectors to make the system effectively three-dimensional, can be presented in the form of three-dimensional gravity coupled  $\sigma$ -model with a symmetric target space. When further reduced to two dimensions via an imposition of an additional Killing vector field, the system becomes integrable by means of the inverse scattering transform method [7] [8] [9]. Similar property is shared by the four-dimensional Einstein-Maxwell theory [2], [10], [11], [12], as well as by some more general gravity-coupled scalar-vector models resulting from supergravities [4], [5].

More recent interest to this subject is related to the search of lower dimensional string models. It turns out to be useful to study exact solutions of the low-energy effective string actions as the first step in the search of exact string backgrounds. Surprisingly, many of exact solutions were obtained in the closed analytical form [13], [14], [15]. Moreover, certain similarity with corresponding solutions to the vacuum Einstein equations can be observed. This fact indicates on the existence of hidden symmetries of lower-dimensional reductions of the theory. It was shown recently that within the context of the Einstein-Maxwell-dilaton-axion theory the hidden symmetry group includes Ehlers-Harrison-type transformations [16], the corresponding two-dimensional truncation leads to integrable theory [17]. It was observed also that purely dilatonic gravity (without an axion) is less symmetric apart from two exceptional values of the dilaton coupling constant. It is the purpose of the present paper to discuss this subject in some details.

We consider an Einstein-Maxwell-dilaton (EMD) system with general dilaton coupling in four space-time dimensions

$$S = \frac{1}{16\pi} \int \left( -R + 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2 \right) \sqrt{-g} \, d^4x, \quad (1.1)$$

where  $\phi$  is the real scalar field (dilaton),  $F = dA$  is the Maxwell two-form,  $\alpha$  is the dilaton coupling constant. This theory was suggested as one of stringy gravity models [15]. It is also interesting as a minimal model continuously interpolating between two highly symmetric systems: Einstein-Maxwell (EM) and five-dimensional Kaluza-Klein (KK) theories.

For  $\alpha = 0$  the action (1.1) describes the EM system with the gravitationally coupled scalar field. It is well-known that the pure EM system becomes an integrable theory provided a two-dimensional Abelian Killing space-time symmetry is imposed. Possible way to demonstrate this property consists in the following. Imposing first one Killing vector field one can dimensionally reduce the system to a gravity coupled three-dimensional sigma-model with the target space  $SU(2, 1)/S(U(1) \times U(2))$  [2],

[11], [12]. Such a system in presence of the second Killing vector field commuting with the first one admits two-dimensional modified chiral matrix representation of Belinskii–Zakharov type [7]. With a scalar field added, the  $\alpha = 0$  action (1.1) can equivalently be thought of as the Brans–Dicke–Maxwell (BDM) model in the Einstein frame (with the Brans–Dicke parameter  $\omega = -1$ ). Obviously this system possesses the same integrability properties as the EM one [18].

Similar integrability property is shared by the KK theory with three commuting Killing vectors, one of which corresponds to the  $x^5$ -translations, [19], [3], [20]. In the adapted system of coordinates this theory can be presented as the four-dimensional EMD system (1.1) with the coupling constant  $\alpha = \sqrt{3}$ . With two commuting Killing symmetries imposed, it reduces to the  $SL(3, R)/SO(3)$  modified chiral matrix model.

Both  $SU(2, 1)$  and  $SL(3, R)$  groups are eight-parametric, so it is natural to investigate a possibility of a deeper link between the stationary EM and KK theories. Clearly, the action (1.1) is the simplest model which ensures a continuous interpolation. Using the real parametrization of the target space corresponding to the stationary reduction of the action (1.1) one can make explicit the relationship between these two structures. From the point of view of the string theory, the distinguished value of the dilaton coupling is  $\alpha = 1$ . So apart from the purely mathematical question about the correspondence between KK and BDM chiral models, it is important to know whether the stringy MD model shares the same integrability property.

The rest of the paper is organized as follows. In Sec. 2 three-dimensional sigma-model representation is derived for the stationary EMD system with an arbitrary dilaton coupling constant. In Sec. 3 we present a detailed investigation of isometries of the corresponding target space. The nature of Killing algebra for non-critical coupling is discussed in Sec. 4. Two particular cases  $\alpha = \sqrt{3}$  and  $\alpha = 0$  are then considered in details using the real target space variables (Sec. 5, 6). For the BDM case a correspondence between the symmetry generators obtained and those known in conventional terms of the Ernst potentials is established (Sec. 6). In Sec. 7 we describe complex discrete transformations of the EMD system similar to Bonnor transformations in the EM theory. They are used to derive new asymptotically flat solutions to dilaton gravity describing dipole field configurations. We conclude with some remarks concerning the nature of the symmetry breaking by a non-critical dilaton.

## 2 Dimensional Reduction.

Assuming four-dimensional metric in (1.1) to admit a time-like Killing vector field, one can write the space-time line element as

$$ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1} h_{ij} dx^i dx^j, \quad (2.1)$$

where the function  $f$ , the one-form  $\omega = \omega_i dx^i$  and the 3-metric  $h_{ij}$  depend only on the space coordinates  $x^i, i = 1, 2, 3$ . It can be easily shown that the corresponding Maxwell field is fully describable in terms of two real-valued functions  $v$  and  $a$  of  $x^i$  exactly as in the case of the pure EM field [21]. Indeed, Maxwell equations and Bianchi identities following from (1.1) read

$$\partial_\nu(\sqrt{-g}e^{-2\alpha\phi}F^{\mu\nu}) = 0, \quad (2.2)$$

$$\partial_\nu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 0, \quad (2.3)$$

where  $\tilde{F}^{\mu\nu} = \frac{1}{2}E^{\mu\nu\lambda\tau}F_{\lambda\tau}$ ,  $E^{\mu\nu\lambda\tau} = \epsilon^{\mu\nu\lambda\tau}/\sqrt{-g}$ . With the assumption of stationarity, the  $\mu = i$  component of (2.3) is satisfied by the substitution

$$F_{i0} = \frac{1}{\sqrt{2}}\partial_i v, \quad (2.4)$$

while the  $\mu = i$  component of the Eq. (2.2) is solved by

$$F^{ij} = \frac{f}{\sqrt{2h}}e^{2\alpha\phi}\epsilon^{ijk}\partial_k a. \quad (2.5)$$

The quantities  $v$  and  $a$  may be regarded as electric and magnetic potentials respectively. The remaining components of the  $F^{\mu\nu}$  can be expressed in terms of  $v$  and  $a$  using the relation [21]

$$F^{i0} = F^{ij}\omega_j - h^{ij}F_{j0}, \quad (2.6)$$

where  $h^{ij}$  is the 3-inverse of  $h_{ij}$ . Another useful relation is

$$F_{ij} = f^{-2}h_{ik}h_{jl}F^{kl} + 2F_{0[i}\omega_{j]}. \quad (2.7)$$

Following Israel and Wilson [21] one can introduce a 3-dual to the rotation 2-form  $d\omega$

$$\tau^i = -f^2\frac{\epsilon^{ijk}}{\sqrt{h}}\partial_j\omega_k, \quad (2.8)$$

which is invariant under the time transformation  $t \rightarrow t + T(x^i)$ . We assume further that the indices of all 3-dimensional quantities are raised and lowered with the 3-metric  $h_{ij}$  while for 4-dimensional tensors one still uses  $g_{\mu\nu}$ . Then the relevant components of the 4-Ricci tensor  $R_{\mu\nu}$  (defined as  $R_{\mu\nu} = \partial_\mu\Gamma_{\nu\alpha}^\alpha - \dots$ ) can be presented as

$$R_{00} = \frac{1}{2}\left(f\Delta f - (\nabla f)^2 + \tau^2\right), \quad (2.9)$$

$$R_0^i = \frac{f}{2\sqrt{h}}\epsilon^{ijk}\tau_{k,j}, \quad (2.10)$$

$$R^{ij} = f^2\Re^{ij} - \frac{1}{2}\left[(\nabla^i f)(\nabla^j f) + \tau^i\tau^j - h^{ij}\left(f\Delta f - (\nabla f)^2 + \tau^2\right)\right], \quad (2.11)$$

where  $\mathfrak{R}^{ij}$  is the Ricci tensor of the 3-space,  $\nabla_i$  denotes 3-covariant derivative,  $\Delta = \nabla^2$ , and 3-vector scalar products are understood with respect to the metric  $h_{ij}$ .

The corresponding components of the stress-energy tensor are:

$$16\pi(T_{00} - \frac{1}{2}g_{00}T) = f \left( (\nabla v)^2 e^{-2\alpha\phi} + (\nabla a)^2 e^{2\alpha\phi} \right), \quad (2.12)$$

$$8\pi T_0^i = \frac{f}{\sqrt{h}} \epsilon^{ijk} (\nabla_j v) (\nabla_k a), \quad (2.13)$$

$$\begin{aligned} 8\pi(T^{ij} - \frac{1}{2}g^{ij}T) = & -f \left( e^{-2\alpha\phi} (\nabla^i v) (\nabla^j v) + e^{2\alpha\phi} (\nabla^i a) (\nabla^j a) \right) \\ & + \frac{fh^{ij}}{2} \left( e^{-2\alpha\phi} (\nabla v)^2 + e^{2\alpha\phi} (\nabla a)^2 \right) + 2f^2 (\nabla^i \phi) (\nabla^j \phi) \end{aligned} \quad (2.14)$$

Note that the dilation does not influence mixed components of the Einstein equations. Hence, comparing (2.10) and (2.13) one obtains

$$\tau_i = w_i + \nabla_i \chi, \quad (2.15)$$

where

$$w_i = v \nabla_i a - a \nabla_i v, \quad (2.16)$$

and  $\chi$  is the twist potential defined up to an additive constant. In terms of  $\chi$  and  $\mathbf{w}$  the 00-component of the Einstein equations will read

$$f \Delta f - (\nabla f)^2 = f \left( (\nabla v)^2 e^{-2\alpha\phi} + (\nabla a)^2 e^{2\alpha\phi} \right) - (\nabla \chi + \mathbf{w})^2. \quad (2.17)$$

The divergence of (2.15) combined with (2.8) leads to the equation for  $\chi$

$$f \Delta \chi - 2 \nabla f (\nabla \chi + \mathbf{w}) + f(v \Delta a - a \Delta v) = 0. \quad (2.18)$$

To obtain second order equations for  $v$  and  $a$  one has to use the  $\mu = 0$  components of the equations (2.2), (2.3). Then taking into account (2.6), (2.7), (2.8) and (2.15) one gets

$$f^2 \nabla (f^{-1} e^{-2\alpha\phi} \nabla v) + (\nabla \chi + \mathbf{w}) \nabla a = 0, \quad (2.19)$$

$$f^2 \nabla (f^{-1} e^{2\alpha\phi} \nabla a) - (\nabla \chi + \mathbf{w}) \nabla v = 0. \quad (2.20)$$

Finally, the dilaton equation in terms of the same variables reads

$$2f \Delta \phi = \alpha \left( e^{-2\alpha\phi} (\nabla v)^2 - e^{2\alpha\phi} (\nabla a)^2 \right). \quad (2.21)$$

The set of equations (2.17)–(2.21) has to be completed by the remaining  $ij$ -Einstein equations. Combining (2.11) and (2.14) and using (2.15) one obtains for the 3-dimensional Ricci tensor the following expression

$$\mathfrak{R}_{ij} = \frac{1}{2f^2} [(\nabla_i f)(\nabla_j f) + (\nabla_i \chi + w_i)(\nabla_j \chi + w_j)] +$$

$$+ 2(\nabla_i \phi)(\nabla_j \phi) - f^{-1} \left[ e^{-2\alpha\phi} (\nabla_i v)(\nabla_j v) + e^{2\alpha\phi} (\nabla_i a)(\nabla_j a) \right]. \quad (2.22)$$

The system (2.17)–(2.22) provides a fully 3-dimensional description of the stationary EMD system with an arbitrary dilaton coupling constant  $\alpha$ . It can be regarded as the 3-dimensional Einstein–matter system with five real scalar fields

$$\varphi^A = (f, \chi, v, a, \phi), \quad A = 1, \dots, 5, \quad (2.23)$$

acting as a source. It can equivalently be derived from the following 3-dimensional gravity coupled  $\sigma$ -model action

$$S_\sigma = \int \left( \mathfrak{R} - \mathcal{G}_{AB}(\varphi) \partial_i \varphi^A \partial_j \varphi^B \right) h^{ij} \sqrt{h} \, d^3 x, \quad (2.24)$$

where  $\mathfrak{R} = \mathfrak{R}_i^i$ , and  $\mathcal{G}_{AB}(\varphi)$  is the target space metric

$$\begin{aligned} \mathcal{G} = \mathcal{G}_{AB} d\varphi^A d\varphi^B = & \frac{1}{2f^2} \left( df^2 + (d\chi + vda - adv)^2 \right) - \\ & - \frac{1}{f} (e^{-2\alpha\phi} dv^2 + e^{2\alpha\phi} da^2) + 2d\phi^2. \end{aligned} \quad (2.25)$$

For  $\alpha = 0$  and  $\phi = \text{const}$  this metric reduces to one given by Neugebauer and Kramer for the EM system [1]. It is worth to be noted that the dilaton equation (2.21) for  $\phi = \text{const}$  gives a constraint on the Maxwell field,  $F^2 = 0$ . Clearly the stationary EMD system may only have  $\phi = \text{const}$  solutions if  $F^2 = 0$ . Hence, the EM system *is not* a particular case of the EMD system. Rather, when we put  $\alpha = 0$ , we get the BMD system, this case will be discussed in details in the Sec. 6.

This representation of the stationary EMD system will be the starting point for the subsequent investigation of the associated hidden symmetries. Some of them can be readily found from the explicit expression (2.25) for the target space metric. However, it turns out that the number of symmetry generators depends on the value of the dilaton coupling constant  $\alpha$  in a somewhat tricky way. One needs to undertake the complete analysis of the target space isometries in order to understand mutual relationship between different symmetry groups arising for two critical values of  $\alpha$ .

### 3 Integration of the target space Killing equations

It is convenient to introduce instead of  $f$  and  $\phi$  the following new variables

$$\eta = \alpha\phi - \frac{1}{2} \ln f, \quad \xi = -(\alpha\phi + \frac{1}{2} \ln f), \quad (3.1)$$

and to define parameters

$$p = \frac{\alpha^2 + 1}{2\alpha^2}, \quad q = \frac{\alpha^2 - 1}{2\alpha^2}, \quad (3.2)$$

provided  $\alpha \neq 0$  (the case  $\alpha = 0$  will be discussed separately). Then the target space metric (2.25) takes the simple form

$$\mathcal{G} = p(d\eta^2 + d\xi^2) + 2qd\eta d\xi - e^{2\xi}dv^2 - e^{2\eta}da^2 + \frac{1}{2}e^{2(\eta+\xi)}(d\chi + vda - adv)^2. \quad (3.3)$$

Our aim is to find all isometries of the target space, that is to construct a complete set of solutions to the Killing equations

$$X_{A;B} + X_{B;A} = 0, \quad (3.4)$$

where covariant derivatives refer to the metric  $\mathcal{G}$ . Contracting (3.4) with  $d\varphi^A d\varphi^B$  and substituting (3.3) one obtains the following equation in terms of bilinear forms:

$$\begin{aligned} & \frac{1}{2}(d\chi + vda - adv)[(X^\eta + X^\xi)(d\chi + vda - adv) + X^v da - X^a dv + dX^\chi \\ & + vdX^a - adX^v]e^{2(\eta+\xi)} + p(dX^\eta d\eta + dX^\xi d\xi) + q(dX^\xi d\eta + dX^\eta d\xi) - \\ & - e^{2\eta}(X^\eta da + dX^a)da - e^{2\xi}(X^\xi dv + dX^v)dv = 0. \end{aligned} \quad (3.5)$$

It contains a set of 15 independent equations, which can be solved by extracting the explicit dependence of  $X^A$  on the field variables  $(\eta, \xi, \chi, v, a)$  step by step. Collecting the  $d\eta^2$  and  $d\xi^2$  terms in (3.5), one obtains two equations

$$(pX^\eta + qX^\xi)_{,\eta} = 0 \quad , \quad (pX^\xi + qX^\eta)_{,\xi} = 0, \quad (3.6)$$

which can be solved

$$X^\eta = \frac{p\pi - q\kappa}{p - q}, \quad X^\xi = \frac{p\kappa - q\pi}{p - q}, \quad (3.7)$$

in terms of two functions of 4 variables  $\pi = \pi(\xi, \chi, v, a)$  and  $\kappa = \kappa(\eta, \chi, v, a)$ . Then from the  $d\xi d\eta$  equation

$$p(X^\eta_{,\xi} + X^\xi_{,\eta}) + q(X^\xi_{,\xi} + X^\eta_{,\eta}) = 0,$$

one can find an explicit dependence of  $\pi$  and  $\kappa$  on  $\xi$  and  $\eta$

$$\begin{aligned} \pi &= A(\chi, v, a)\xi + B(\chi, v, a), \\ \kappa &= -A(\chi, v, a)\eta + C(\chi, v, a). \end{aligned} \quad (3.8)$$

(Here and in what follows capital Latin letters denote the differentiable functions of variables indicated in the parenthesis.)

To extract the  $\chi$ -dependence we use the  $d\eta d\chi$  and  $d\xi d\chi$  equations

$$2(pX^\eta + qX^\xi)_{,\chi} + e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\eta} = 0, \quad (3.9)$$

$$2(pX^\xi + qX^\eta)_{,\chi} + e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\xi} = 0, \quad (3.10)$$

yielding

$$\begin{aligned} A &= A(v, a), \quad C = B + F(v, a), \\ X^\chi + vX^a - aX^v &= e^{-2(\eta+\xi)} B_{,\chi} + D(\chi, v, a), \end{aligned} \quad (3.11)$$

together with the  $d\chi^2$  equation

$$X^\eta + X^\xi + (X^\chi + vX^a - aX^v)_{,\chi} = 0. \quad (3.12)$$

They imply

$$\begin{aligned} A &\equiv 0, \quad B = G(v, a)\chi + H(v, a), \\ D &= -G(v, a)\chi^2 - (2H(v, a) + F(v, a))\chi + I(v, a). \end{aligned} \quad (3.13)$$

The next step of the derivation is the  $a, v$ -reduction. From the  $dad\eta$  term in (3.5) one gets

$$p\chi_{,a}^\eta + \frac{v}{2}e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\eta} - e^{2\eta}X_{,\eta}^a + qX_{,a}^\xi = 0. \quad (3.14)$$

From here, with account for (3.10) and (3.13), one arrives at

$$X^a = -\frac{1}{2}e^{-2\eta}(G_{,a}\chi + H_{,a} - Gv) + K(\xi, \chi, v, a). \quad (3.15)$$

Then, from the  $dad\xi$  equation

$$(pX^\xi + qX^\eta)_{,a} + \frac{v}{2}e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\xi} - e^{2\eta}X_{,\xi}^a = 0 \quad (3.16)$$

the function  $K$  is seen to be independent on  $\xi$ ,  $G = G(v)$  and also

$$H + F = G(v)av + L(v). \quad (3.17)$$

Similarly, using the  $dvd\xi$  equation

$$pX_{,v}^\xi - \frac{a}{2}e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\xi} - e^{2\xi}X_{,\xi}^v + qX_{,v}^\eta = 0, \quad (3.18)$$

we find

$$X^v = -\frac{1}{2}e^{-2\xi}(G_{,v}\chi + H_{,v} + F_{,v} + Ga) + M(\eta, \chi, v, a). \quad (3.19)$$

Entering with (3.19) into the  $dvd\eta$  equation one gets

$$pX_{,v}^\eta - \frac{a}{2}e^{2(\eta+\xi)}(X^\chi + vX^a - aX^v)_{,\eta} - e^{2\xi}X_{,\eta}^v + qX_{,v}^\xi = 0. \quad (3.20)$$



From here it can be derived

$$\begin{aligned} G &= \text{const}, \quad H = -Gva + N(a), \quad M = M(\chi, v, a), \\ F &= 2Gva + L(v) - N(a). \end{aligned} \quad (3.21)$$

Collecting all previous results one can write the general solution of the Killing equations as follows

$$\begin{aligned} X^\eta &= G\chi + (pN(a) - qL(v) - Gva)(p - q)^{-1}, \\ X^\xi &= G\chi + (pL(v) - qN(a) + Gva)(p - q)^{-1}, \\ X^v &= -\frac{1}{2}e^{-2\xi}(2Ga + L'(v)) + M(\chi, v, a), \\ X^a &= \frac{1}{2}e^{-2\eta}(2Gv - N'(a)) + K(\chi, v, a), \end{aligned} \quad (3.22)$$

$$X^\chi + vX^a - aX^v = G(e^{-2(\eta+\xi)} - \chi^2) - (N(a) + L(v))\chi + I(v, a).$$

with  $G = \text{const}$ , where primes stand for the derivatives with respect to the corresponding single argument.

For a subsequent reduction one uses first the  $dad\chi$  equation obtaining the relations

$$\begin{aligned} K &= -\frac{1}{2}(L'(v) + 2Ga)\chi + P(v, a), \\ M &= \frac{1}{2}(N'(a) - 2Gv)\chi - \frac{1}{2}[I_{,a} + v(N + L)], \end{aligned} \quad (3.23)$$

and then the  $dvd\chi$  equation giving

$$\begin{aligned} M &= \frac{1}{2}(N'(a) - 2Gv)\chi + Q(v, a), \\ K &= -\frac{1}{2}(L'(v) + 2Ga)\chi + \frac{1}{2}(I_{,v} - a(N + L)). \end{aligned} \quad (3.24)$$

Combining (3.23) and (3.24) one gets

$$\begin{aligned} P(v, a) &= \frac{1}{2}[I_{,v} - a(N + L)], \\ G(v, a) &= -\frac{1}{2}[I_{,a} + v(N + L)]. \end{aligned} \quad (3.25)$$

Furthermore, the  $da^2$  equation yields  $N = Ra + S$  with constant  $R$  and  $S$ , and

$$I_{,va} = (p - q)^{-1}(L - N + 4qGav) + aN' - vL'. \quad (3.26)$$

Similarly, from the  $dv^2$  equation one gets  $L = Tv + U$ , with constant  $T$  and  $U$ , as well as (3.26) again. Finally, the crossed term  $dvda$  in (3.5) gives two additional equations for  $I(v, a)$

$$I_{,vv} = 2a(Ga + T), \quad (3.27)$$

$$I_{,aa} = 2v(Gv - R). \quad (3.28)$$

Differentiating (3.26) over  $v$  and (3.27) over  $a$  we obtain the following consistency condition:

$$\frac{q}{p-q}(T + 2Ga) = T + 2Ga. \quad (3.29)$$

In the same way, differentiating (3.26) over  $a$  and (3.28) over  $v$  one gets

$$\frac{q}{p-q}(2Gv - R) = 2Gv - R. \quad (3.30)$$

Now it is clear from (3.29) and (3.30), that the constants  $G, R, T$  can be non-zero if and only if

$$\frac{q}{p-q} = 1, \quad (3.31)$$

what, on account for (3.2), means

$$\alpha = \sqrt{3}. \quad (3.32)$$

For all other (non-zero) values of  $\alpha$  one has  $G = R = T = 0$ , and the solution of (3.26)–(3.28) will read

$$I = \frac{1}{p-q}(U - S)av + Wa + Vv + Z, \quad (3.33)$$

where  $W, V$  and  $Z$  are constants, and  $N = S, L = U$ . Therefore for an arbitrary value of  $\alpha$ , except for  $\alpha = 0$  and  $\alpha = \sqrt{3}$ , the general solution of the Killing equations obtained by substituting (3.24), (3.25), (3.33) into (3.22) reads

$$\begin{aligned} X = X^A \frac{\partial}{\partial \varphi^A} = & S(p-q)^{-1} [p(\partial_\eta - a\partial_a) + q(v\partial_v - \partial_\xi)] - S\chi\partial_\chi + \\ & + U(p-q)^{-1} [p(\partial_\xi - v\partial_v) + q(a\partial_a - \partial_\eta) - \chi\partial_\chi] + \frac{1}{2}V(\partial_a + v\partial_\chi) + \\ & + \frac{1}{2}W(a\partial_\chi - \partial_v) + Z\partial_\chi. \end{aligned} \quad (3.34)$$

It describes five-parametric isometry group of the target space for any non-critical  $\alpha$ . For  $\alpha = 0$  the metric representation (3.3) is not valid, this case will be discussed later in the Sec. 6. For  $\alpha = \sqrt{3}$  three additional Killing vectors arise which correspond to the non-zero constants  $G, R, T$ .

## 4 Symmetry algebra for a non-critical coupling

It is convenient to choose five independent Killing vectors as follows:

$$\begin{aligned} X_1 &= \partial_\eta - a\partial_a - \chi\partial_\chi, \\ X_2 &= \partial_\xi - v\partial_v - \chi\partial_\chi, \\ X_3 &= \partial_a + v\partial_\chi, \\ X_4 &= \partial_v - a\partial_\chi, \\ X_5 &= 2\partial_\chi, \end{aligned} \tag{4.1}$$

where linear combinations of  $S$  and  $U$  used in (3.34) to get  $X_1$  and  $X_2$  symmetric under the interchange  $\eta \leftrightarrow \xi$ .  $a \leftrightarrow v$ . Note that the discrete duality transformation

$$\eta \leftrightarrow \xi, \quad a \leftrightarrow v, \quad \chi \leftrightarrow -\chi, \tag{4.2}$$

which is obviously the symmetry of the target space metric (3.3), leaves the whole set (4.1) invariant.

Five Killing vectors (4.1) form a closed 5-dimensional algebra  $X$

$$[X_\mu, X_\nu] = C^\lambda_{\mu\nu} X_\lambda, \quad \mu, \nu, \lambda = 1, \dots, 5, \tag{4.3}$$

with the following non-zero structure constants:

$$C^3_{13} = C^4_{24} = C^5_{15} = C^5_{25} = -C^5_{34} = 1 \tag{4.4}$$

This algebra is solvable. Indeed, its derivative  $X'$  contains as basis vectors  $X_3, X_4, X_5$ , the second derivative is one-dimensional  $X'' = X_5$ , and we have the following chain of subalgebras

$$0 = X''' \subset X'' \subset X' \subset X, \tag{4.5}$$

where all terms are the subsequent ideals of the previous ones.

Using (4.4) it can be shown that the Killing-Cartan bilinear form

$$C^\alpha_{\mu\beta} C^\beta_{\nu\alpha} \Omega^\mu \Omega^\nu = 2(\Omega^1)^2 + 2\Omega^1 \Omega^2 + (\Omega^2)^2 \tag{4.6}$$

is non-degenerate only on 1-2 subspace.

Such algebras are known to admit a representation in terms of the upper triangle matrices. Consistently with (4.2)  $X_1$  and  $X_2$  can be choosen diagonal. Then the following  $3 \times 3$  representation  $X_\mu \rightarrow e_\mu$  holds

$$e_1 = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

Obviously, this set constitutes a basis for the upper triangle subalgebra of  $sl(3, R)$ .

From (4.6) it is clear that, the algebra of Killing vectors (4.3) is insufficient to provide the target space with a symmetric Riemannian space structure. From 5 generators (4.1) 3 express pure gauge degrees of freedom ( $X_3$  and  $X_4$ —electromagnetic,  $X_5$ —gravitational). The corresponding finite transformations read respectively:

$$a \rightarrow a + \lambda_3, \quad \chi \rightarrow \chi + \lambda_3 v, \quad (4.8)$$

$$v \rightarrow v + \lambda_4, \quad \chi \rightarrow \chi - \lambda_4 a, \quad (4.9)$$

$$\chi \rightarrow \chi + \lambda_5, \quad (4.10)$$

where  $\lambda_3, \lambda_4, \lambda_5$  are the real group parameters. Two other transformations ( $X_1, X_2$ ) are the linear combinations of a dilaton constant shift (accompanied by a suitable rescaling of  $v$  and  $a$ ) and a scale transformation:

$$\eta \rightarrow \eta + \lambda_1 \quad a \rightarrow ae^{-\lambda_1}, \quad \chi \rightarrow \chi e^{-\lambda_1}, \quad (4.11)$$

$$\xi \rightarrow \xi + \lambda_2, \quad v \rightarrow ve^{-\lambda_2}, \quad \chi \rightarrow \chi e^{-\lambda_2}. \quad (4.12)$$

Comparing this with the EM case [2] we see that the Ehlers–Harrison part of the symmetry group is lacking. This destroys the integrability of the EM equations in presence of the second space–like Killing vector field commuting with the initial time–like one is imposed. Integrability property is, however, restored for a particular value of the dilaton coupling constant  $\alpha = \sqrt{3}$ , when the matrix algebra (4.7) enlarges to the full  $sl(3, R)$ .

## 5 Kaluza–Klein theory

In this exceptional case the consistency conditions (3.29) and (3.30) are fulfilled identically and the solution of the equations (3.26)–(3.28) reads

$$I = 3(U - S)av + av(Tv - Ra + Gav) + Wa + Vv + Z. \quad (5.1)$$

Thus, the general solution of the Killing equations contains 3 additional real parameters:  $G, R$  and  $T$ . Substituting (4.1) into (3.23)–(3.25) and further to (3.22), we get the 8–parametric solution. In addition to five ( $\alpha$ –independent) Killing vectors (3.35) now we have

$$X_6 = v(\partial_\eta - 2\partial_\xi) + \left(\frac{1}{2}e^{-2\xi} + v^2\right)\partial_v + \frac{1}{2}\chi_1\partial_a + \frac{1}{2}(ae^{-2\xi} + v\chi_{-1})\partial_\chi, \quad (5.2)$$

$$X_7 = a(\partial_\xi - 2\partial_\eta) + \left(\frac{1}{2}e^{-2\eta} + a^2\right)\partial_a - \frac{1}{2}\chi_{-1}\partial_v + \frac{1}{2}(a\chi_1 - ve^{-2\eta})\partial_\chi, \quad (5.3)$$

$$X_8 = -\frac{1}{2}(\chi_3\partial_\eta + \chi_{-3}\partial_\xi) + \frac{1}{2}(v\chi_{-1} + ae^{-2\xi})\partial_v + \frac{1}{2}(a\chi_1 - ve^{-2\eta})\partial_a + \\ + \frac{1}{2}(a^2e^{-2\xi} + v^2e^{-2\eta} + a^2v^2 + \chi^2 - e^{-2(\xi+\eta)})\partial_\chi, \quad (5.4)$$

where  $\chi_n \equiv \chi - nav$ . These generators were found by Neugebauer in 1969 [19] in slightly different variables<sup>3</sup>. Together with (4.1) they form the  $sl(3, R)$  algebra as it can be seen from the commutation relations (4.3), now with  $\mu, \nu, \lambda = 1, \dots, 8$ . The set of non-zero structure constants (4.4) is enlarged to include

$$C^7_{17} = C^8_{18} = C^6_{26} = C^8_{28} = -C^6_{38} = C^7_{48} = -C^3_{56} = C^4_{57} = \\ = -C^8_{67} = -C^2_{37} = -C^1_{46} = C^1_{58} = C^2_{58} = -1, \\ C^1_{37} = C^2_{46} = -2. \quad (5.5)$$

New generators (5.2)–(5.4) have the following  $3 \times 3$  matrix counterparts

$$e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad (5.6)$$

completing the algebra of  $sl(3, R)$ .

The  $sl(3, R)$  symmetry of the 2-stationary (admitting 2 commuting Killing vectors) 5-dimensional KK-theory was first discovered by Maison [3] using another formulation (see also [20]). The remarkable property of the target space (3.1) in the case  $\alpha^2 = 3$  is that it is a homogeneous symmetric Riemannian space on which the group  $SL(3, R)$  acts transitively. It can be checked by a direct computation that the Riemann tensor corresponding to the metric (3.1) for  $p = 2q = 2/3$  is covariantly constant:  $\nabla_A R^B_{CDE} = 0$ . Furthermore, the target space is an Einstein space

$$R_{AB} = \frac{R}{5}\mathcal{G}_{AB}, \quad (5.7)$$

where the scalar curvature  $R = -15$ . Maison [3] showed that this space can be identified with the coset  $SL(3, R)/SO(3)$ . Within the present framework this can be seen as follows.

The Killing–Cartan metric constructed with the structure constants (4.4) and (5.5),

$$\eta_{\mu\nu} = \frac{1}{12}C^\alpha_{\mu\beta}C^\beta_{\nu\alpha} = \frac{1}{2}Tr(e_\mu e_\nu) \quad (5.8)$$

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<sup>3</sup>The authors are grateful to Dr. T. Matos for this reference.

has the following non-zero components:

$$\eta_{11} = \eta_{22} = 2\eta_{12} = \frac{1}{3}, \quad \eta_{37} = \eta_{46} = \eta_{58} = -\frac{1}{2}. \quad (5.9)$$

For the corresponding inverse tensor one finds

$$\eta^{11} = \eta^{22} = -2\eta^{12} = 4, \quad \eta^{37} = \eta^{46} = \eta^{58} = -2. \quad (5.10)$$

Using these quantities, one can build the Killing one-forms

$$\tau^\mu = \eta^{\mu\nu} X_\nu^A \mathcal{G}_{AB} d\varphi^B, \quad (5.11)$$

which satisfy Maurer–Cartan equation

$$d\tau^\mu + \frac{1}{2} C^\mu_{\alpha\beta} \tau^\alpha \wedge \tau^\beta = 0. \quad (5.12)$$

Explicitly,  $\tau^\mu$  from (5.11) read

$$\begin{aligned} \tau^1 &= 2d\eta + 4ae^{2\eta}da - 2ve^{2\xi}dv + \frac{1}{2}\chi_3\tau^8, \\ \tau^2 &= 2d\xi + 4ve^{2\xi}dv - 2ae^{2\eta}da + \frac{1}{2}\chi_{-3}\tau^8, \\ \tau^3 &= 2ad\eta + (2a^2e^{2\eta} + 1)da - \chi_{-1}e^{2\xi}dv + \frac{a}{2}\chi_{-1}\tau^8, \\ \tau^4 &= 2vd\xi + (2v^2e^{2\xi} + 1)dv + \chi_1e^{2\eta}da + \frac{v}{2}\chi_1\tau^8, \\ \tau^5 &= \chi_1d\eta + \chi_{-1}d\xi + (a + v\chi_{-1}e^{2\xi})dv - \\ &\quad -(v - a\chi_1e^{2\eta})da - \frac{1}{4}(a^2v^2 - \chi^2 + e^{-2(\xi+\eta)})\tau^8, \\ \tau^6 &= 2e^{2\xi}dv - a\tau^8, \\ \tau^7 &= 2e^{2\eta}da + v\tau^8, \\ \tau^8 &= -2e^{2(\xi+\eta)}(d\chi + vda - adv). \end{aligned} \quad (5.13)$$

In terms of these Killing one-forms the line element of the target space (3.3) with  $p = 2q = 2/3$  can be written as

$$\begin{aligned} \mathcal{G} &= \frac{1}{2}\eta_{\mu\nu}\tau^\mu \otimes \tau^\nu = \frac{1}{6}(\tau^1 \otimes \tau^1 + \tau^2 \otimes \tau^2 + \tau^1 \otimes \tau^2) - \\ &\quad - \frac{1}{2}(\tau^3 \otimes \tau^7 + \tau^4 \otimes \tau^6 + \tau^5 \otimes \tau^8), \end{aligned} \quad (5.14)$$

or, alternatively, (see (5.7))

$$\mathcal{G} = \frac{1}{4} \text{Tr}(\mathbf{A} \otimes \mathbf{A}), \quad (5.15)$$

where the  $3 \times 3$  matrix one-form  $\mathbf{A}$  is defined as

$$\mathbf{A} = A_B d\varphi^B = e_\mu \tau^\mu \quad (5.16)$$

In view of (5.11) the one-form  $\mathbf{A}$  has a vanishing curvature

$$F_{BC} = A_{C,B} - A_{B,C} + [A_B, A_C] = 0, \quad (5.17)$$

and thus  $A_B$  is a pure gauge

$$A_B = (\partial_B U) U^{-1}, \quad (5.18)$$

where  $U$  is a  $3 \times 3$  matrix. To find it explicitly, it is convenient to use Gauss decomposition of the general  $SL(3, R)$  matrix

$$M = M_L M_D M_R, \quad (5.19)$$

where  $M_R, M_L$  are right- (left-) triangle matrices and  $M_D$  is diagonal (all with unit determinant)

$$M_R = \begin{pmatrix} 1 & r_1 & r_3 \\ 0 & 1 & r_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & (d_1 d_2)^{-1} \end{pmatrix}, \quad M_L = \begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_3 & l_2 & 1 \end{pmatrix}. \quad (5.20)$$

The corresponding generators are given by the sets  $(e_3, e_4, e_5)$ ,  $(e_1, e_2)$  and  $(e_6, e_7, e_8)$  respectively.

For any  $U \in SL(3, R)/SO(3)$  one has  $U_L = U_R^T$ , so using a parametrization

$$U_R = \begin{pmatrix} 1 & p_1 & p_3 \\ 0 & 1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_D = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & (q_1 q_2)^{-1} \end{pmatrix}, \quad (5.21)$$

we obtain

$$U = \begin{pmatrix} q_1 & q_1 p_1 & q_1 p_3 \\ q_1 p_1 & q_2 + q_1 p_1^2 & q_2 p_2 + q_1 p_1 p_3 \\ q_1 p_3 & q_2 p_2 + q_1 p_1 p_3 & q_1 p_3^2 + q_2 p_2^2 + (q_1 q_2)^{-1} \end{pmatrix}. \quad (5.22)$$

Infinitesimal  $SL(3, R)$  transformations

$$U \rightarrow M^T U M, \quad M = I + \lambda^\mu \hat{X}_\mu \quad (5.23)$$

lead to the following representation of generators in terms of the  $q, p$ -derivatives:

$$\hat{X}_1 = \frac{1}{3} (4q_1 \partial_{q_1} - 3p_1 \partial_{p_1} - 2q_2 \partial_{q_2} - 3p_3 \partial_{p_3}),$$

$$\begin{aligned}
\hat{X}_2 &= \frac{1}{3}(2q_1\partial_{q_1} - 3p_2\partial_{p_2} + 2q_2\partial_{q_2} - 3p_3\partial_{p_3}), \\
\hat{X}_3 &= \partial_{p_1}, \\
\hat{X}_4 &= -p_1\partial_{p_3} - \partial_{p_2}, \\
\hat{X}_5 &= \partial_{p_3}, \\
\hat{X}_6 &= 2q_2p_2\partial_{q_2} + p_3\partial_{p_1} + (q_1^{-1}q_2^{-1} - p_2^2)\partial_{p_2}, \\
\hat{X}_7 &= -2q_1p_1\partial_{q_1} + 2q_2p_1\partial_{q_2} - (q_1^{-1}q_2 - p_1^2)\partial_{p_1} + (p_3 - p_1p_2)\partial_{p_2} + (p_1p_3 - q_2p_2q_1^{-1})\partial_{p_3}, \\
\hat{X}_8 &= -2q_1p_3\partial_{q_1} + 2q_2p_1p_2\partial_{q_2} + (p_1p_3 - q_2p_2q_1^{-1})\partial_{p_1} + \\
&\quad (p_2p_3 - p_1p_2^2 + p_1q_1^{-1}q_2^{-2})\partial_{p_2} + (p_3^2 - q_2p_2^2q_1^{-1} - q_1^{-2}q_2^{-1})\partial_{p_3},
\end{aligned} \tag{5.24}$$

Comparing this with our initial representation (4.1), (5.2), (5.3), (5.4) in terms of the  $\sigma$ -model variables, one finds

$$q_1 = -2e^{2(\xi+2\eta)/3}, \quad q_2 = e^{2(\xi-\eta)/3}, \quad p_1 = u, \quad p_2 = -v, \quad p_3 = (\chi - uv)/2. \tag{5.25}$$

As a result we obtain the following representation for the matrix  $U \in SL(3, R)/SO(3)$ :

$$\begin{aligned}
U &= -\mu \begin{pmatrix} 2 & 2a & \chi - av \\ 2a & 2a^2 - e^{-2\eta} & a(\chi - av) + ve^{-2\eta} \\ \chi - av & a(\chi - av) + ve^{-2\eta} & ((\chi - av)^2 - 2v^2e^{-2\eta} + e^{-2(\xi+\eta)})/2 \end{pmatrix}, \\
\mu &= \exp \left[ \frac{1}{3}(2\xi + 4\eta) \right].
\end{aligned} \tag{5.26}$$

(This matrix is related to one found by Maison [3] by some constant  $SL(3, R)$  transformation.)

Using (5.15) and (5.18) we obtain the metric of the target space in terms of the coset  $SL(3, R)/SO(3)$  variables, namely

$$\mathcal{G} = \frac{1}{4}Tr(dUU^{-1} \otimes dUU^{-1}). \tag{5.27}$$

The equation of motion in these variables can be derived directly from the action, or through the following argument [12]. The pull-back of the Killing one-form onto the configuration space constitutes the set of Noether currents

$$J_i^\mu = \tau_A^\mu \frac{\partial \varphi^A}{\partial x^i} \tag{5.28}$$

which are conserved in view of the invariance of the action under  $SL(3, R)$ :

$$\partial_i(h^{ij}\sqrt{h}J_i^\mu) = 0. \tag{5.29}$$



As it is known, for a symmetric target space such a set of conservation laws is equivalent to the equations of motion. Introducing the pull-back  $\mathcal{A} = \mathcal{A}_i dx^i$  of  $\mathbf{A}$ , where  $\mathcal{A}_i = A_B \partial \varphi^B / \partial x^i$ , we can rewrite (5.22) as

$$d \star \mathcal{A} = 0, \quad (5.30)$$

or

$$d \left( \star d U U^{-1} \right) = 0, \quad (5.31)$$

where the star stands for the 3-dimensional Hodge dual operation. This form of the field equations is suitable for an application of the inverse scattering transform method in the axisymmetric case [7], [12]. An alternative development following the prolongation structure technique can be found in [20]. Another derivation of Belinskii–Zakharov type of a system was given in [9] using different reduction scheme from the 5-dimensional KK theory. Our purpose here was to demonstrate explicitly how the  $SL(3, R)$  symmetry, generally broken by the non-critically coupled dilaton, turns out to be restored for the critical coupling  $\alpha = \sqrt{3}$ . Note, that another particular value of this coupling constant,  $\alpha = 1$ , which emerges in the context of the heterotic string low-energy effective theory, belongs to the broken symmetry case.

## 6 Brans–Dicke–Maxwell model, $\alpha = 0$

For  $\alpha = 0$  the parametrization (3.3) of the target space metric is not valid, and we have to restart with the initial parametrization (2.25). Obviously, the dilaton decouples from the Einstein–Maxwell part of the  $\sigma$ -model, which is described here in Neugebauer and Kramer variables [1]. As it is clear from (1.1), for  $\alpha = 0$  the system (including the dilaton) is just the BDM system with purely gravitationally coupled scalar field (in the Einstein frame). For completeness we present here the corresponding Killing algebra in terms of real variables and then translate it into the standard Ernst-potentials form.

Three of five Killing vectors (3.35) from the maximal solvable subalgebra of  $sl(3, R)$ , remain Killing vectors in the  $\alpha = 0$  case:  $X_3, X_4, X_5$  (the electromagnetic and gravitational gauge transformations). The sum  $X_1 + X_2$  is dilaton-independent and hence remains the symmetry too (the scale transformation). The difference  $X_1 - X_2$  in the limit  $\alpha \rightarrow 0$  reduces to the pure dilaton shift. In addition, a continuous duality rotation emerges as a symmetry (it is broken by the dilaton for any  $\alpha \neq 0$ ). The non-trivial part of the target space isometry algebra consists of the Harrison transformations and the Ehlers transformation. Altogether one has the

9-dimensional algebra of isometries generated by the following set of Killing vectors:

$$\begin{aligned}
\bar{X}_1 &= v\partial_a - a\partial_v, \\
\bar{X}_2 &= -(X_1 + X_2) = 2(f\partial_f + \chi\partial_\chi) + a\partial_a + v\partial_v, \\
\bar{X}_3 &= X_3, \quad \bar{X}_4 = X_4, \quad \bar{X}_5 = X_5, \\
\bar{X}_6 &= 2fv\partial_f + (v\chi + aF - )\partial_\chi + \left(\frac{1}{2}(v^2 - 3a^2) + f\right)\partial_v + \chi_{-2}\partial_a, \\
\bar{X}_7 &= 2fa\partial_f + (a\chi - vF)\partial_f + \left(\frac{1}{2}(a^2 - 3v^2) + f\right)\partial_a - \chi_2\partial_v, \\
\bar{X}_8 &= 2f\chi\partial_f + (\chi^2 - F^2)\partial_\chi + (v\chi - aF)\partial_v + (a\chi + vF)\partial_a, \\
\bar{X}_9 &= \partial_\phi,
\end{aligned} \tag{6.1}$$

where  $F = f - (v^2 + a^2)/2$ . Obviously  $\bar{X}_9$  commutes with all other generators, while the remaining non-zero structure constants read

$$\begin{aligned}
C^4_{13} &= -C^3_{14} = -C^7_{16} = C^6_{17} = -C^3_{23} = -C^4_{24} = C^6_{26} = \\
&= C^7_{27} = -C^5_{34} = C^2_{37} = C^6_{38} = C^2_{46} = -C^7_{48} = 1, \\
-C^5_{25} &= C^8_{28} = C^3_{56} = -C^4_{57} = C^2_{58} = C^8_{67} = 2, \\
C^1_{36} &= -C^1_{47} = 3.
\end{aligned} \tag{6.2}$$

The structure constants (6.2) form a  $su(2, 1)$  algebra. To cast it into more conventional form one has to introduce complex Ernst potentials

$$\varepsilon = f + i\chi - \frac{v^2 + a^2}{2}, \quad \Phi = \frac{v + ia}{\sqrt{2}}. \tag{6.3}$$

For  $\alpha = 0$  the metric (2.25) in terms of these variables will read

$$\mathcal{G} = \frac{1}{2f^2} |d\varepsilon + 2\Phi^* d\Phi|^2 - \frac{2}{f} d\Phi d\Phi^* + 2d\phi^2. \tag{6.4}$$

The complete isometry algebra of (6.4) is isomorphic to  $su(2, 1) \times R$ . In terms of

the complex variables the generators (6.1) take the following form

$$\begin{aligned}
\bar{X}_1 &= i\Phi\partial_\Phi + c.c., \\
\bar{X}_2 &= 2\varepsilon\partial_\varepsilon + \Phi\partial_\Phi + c.c., \\
\bar{X}_3 &= i\left(\frac{1}{\sqrt{2}}\partial_\Phi + \sqrt{2}\Phi\partial_\varepsilon\right) + c.c., \\
\bar{X}_4 &= \frac{1}{\sqrt{2}}\partial_\Phi - \sqrt{2}\Phi\partial_\varepsilon + c.c., \\
\bar{X}_5 &= 2i\partial_\varepsilon + c.c., \\
\bar{X}_6 &= \sqrt{2}\varepsilon\Phi\partial_\varepsilon + \frac{1}{\sqrt{2}}(\varepsilon + 2\Phi^2)\partial_\Phi + c.c., \\
\bar{X}_7 &= -i\sqrt{2}\varepsilon\Phi\partial_\varepsilon + \frac{i}{\sqrt{2}}(\varepsilon - 2\Phi^2)\partial_\Phi + c.c., \\
\bar{X}_8 &= -i\varepsilon(\Phi\partial_\Phi + \varepsilon\partial_\varepsilon) + c.c..
\end{aligned} \tag{6.5}$$

Up to normalization, these generators coincide with ones given previously by Neugebauer [19], and Eris, Gürses and Karasu [12]. Contrary to the KK theory, which has natural description in terms of the real target space variables, for the EM system more appropriate are the complex variables which are intrinsically related to nature of the symmetry group. It is interesting that inspite of this difference, there is a striking similarity between non-trivial sectors of the  $sl(3, R)$  algebra discussed in the previous section ( $X_6, X_7, X_8$ ) and  $su(2, 1)$  algebra here ( $\bar{X}_6, \bar{X}_7, \bar{X}_8$ ). One can easily see this taking the corresponding commutators.

Five-dimensional target space for the BDM stationary system is the product of a symmetric space  $SU(2, 1)/S(U(1) \times U(2))$  and a line  $R$ . Hence the integrability arguments developed for EM system equally apply to the present case. Nevertheless, both theories are physically different. It is worth to be reminded once again that the EM theory *is not* a particular case of the EMD system since the dilaton equation (2.21) imposes a constraint  $F^2 = 0$  on the Maxwell field if only we put  $\phi = const$ .

## 7 Solution-generating technique for non-critical coupling

From the above analysis it is clear that the stationary EMD system with  $\alpha \neq 0, \sqrt{3}$  does not possess enough symmetries to ensure the existence of a full-scale solution generating technique similar to that of the EM and KK theories. Nevertheless there are some non-trivial maps between subspaces of the target space which can be used

to generate new solutions. These maps typically are discrete or present some combinations of discrete maps and continuous transformations described above. From five continuous transformations two are physically meaningful: a constant dilaton shift, accompanied by suitable rescaling of electromagnetic potentials,

$$\begin{aligned}\phi_1 &\rightarrow \phi_2 = \phi_1 + \phi_{1c}, \\ v_1 &\rightarrow v_2 = e^{\alpha\phi_{1c}} v_1, \\ a_1 &\rightarrow a_2 = e^{-\alpha\phi_{1c}} a_1,\end{aligned}\tag{7.1}$$

where  $\phi_{1c} = \text{const.}$ , and the scale transformation

$$\begin{aligned}f_1 &\rightarrow f_2 = C^2 f_1, \quad \chi_1 \rightarrow \chi_2 = C^2 \chi_1, \\ v_1 &\rightarrow v_2 = C v_1, \quad a_1 \rightarrow a_2 = C a_1\end{aligned}\tag{7.2}$$

with the real constant  $C$ .

There are also some discrete symmetries of the target space valid for all  $\alpha$ . One is discrete electric–magnetic duality

$$\phi \rightarrow -\phi, \quad \chi \rightarrow -\chi, \quad v \leftrightarrow a,\tag{7.3}$$

which obviously leaves the metric (2.25) invariant.

Another discrete transformation is highly non-trivial. It is analogous to the well-known Bonnor transformation for the EM system [22]. To derive it, let us consider two subspaces of the target space: the dilaton vacuum

$$dl_1^2 = \frac{df^2 + d\chi^2}{2f^2} + 2d\phi^2,\tag{7.4}$$

and the static magnetic sector

$$dl_2^2 = \frac{df^2}{2f^2} - \frac{e^{2\alpha\phi} da^2}{f} + 2d\phi^2.\tag{7.5}$$

Under a (complex) transformation

$$\begin{aligned}f_1^2 &= f_2 e^{-2\alpha\phi_2}, \\ \chi_1 &= i \left( \frac{1 + \alpha^2}{2} \right)^{1/2} a_2, \\ \phi_1 &= \frac{1}{2} \left( \phi_2 + \frac{\alpha}{2} f_2 \right),\end{aligned}\tag{7.6}$$

the line element (7.4) maps onto (7.5) up to a constant rescaling

$$dl_1^2 = \left( \frac{1 + \alpha^2}{4} \right) dl_2^2. \quad (7.7)$$

Similarly, an “electric” transformation

$$\begin{aligned} f_1^2 &= f_2 e^{2\alpha\phi_2}, \\ \chi_1 &= i \left( \frac{1 + \alpha^2}{2} \right)^{1/2} v_2, \\ \phi_1 &= \frac{1}{2} \left( \phi_2 - \frac{\alpha}{2} f_2 \right) \end{aligned} \quad (7.8)$$

maps (7.4) onto the line element of the electrostatic subspace

$$dl_2^2 = \frac{df^2}{2f^2} - \frac{e^{-2\alpha\phi} dv^2}{f} + 2d\phi^2, \quad (7.9)$$

according the same rescaling rule (7.7). It is worth noting that, contrary to the isometry transformations discussed in previous sections, as well as to original Bonnor map, here we deal with transformations which conformally rescale the target space metric on a constant factor. This does not change  $\sigma$ -model equations, but does change the *three-metric*. If we restrict ourselves by the axially-symmetric case, and represent the three-metric in the Lewis-Papapetrou form

$$dl^2 = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2, \quad (7.10)$$

from the three-dimensional Einstein equations we get

$$\gamma_2 = \left( \frac{4}{1 + \alpha^2} \right) \gamma_1. \quad (7.11)$$

To apply these transformations for generating purposes one needs to know appropriately complexified seed solutions in the same way as in the original Bonnor case. To ensure desired physical properties of resulting spacetime (e.g. asymptotic flatness) one can use the dilaton shift (7.1) and the scale transformation (7.2). As an example we apply combination of (7.6) and (7.1) to the complexified Kerr solution, which belongs to the subspace (7.4) with  $\phi \equiv 0$ ,

$$ds^2 = \frac{\Delta + b^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi)^2 - \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta + b^2 \sin^2 \theta} d\varphi^2 \right). \quad (7.12)$$

Here

$$\Delta = r^2 - 2mr - b^2, \quad \Sigma = r^2 - b^2 \cos^2 \theta, \quad \omega = -\frac{2imbr \sin^2 \theta}{\Delta + b^2 \sin^2 \theta}, \quad (7.13)$$

and parameter  $b$  is related to the original Kerr rotation parameter  $a$  as  $b = ia$ . The corresponding twist potential is pure imaginary

$$\chi = 2imb \cos \theta \Sigma^{-1}, \quad (7.14)$$

and the metric function  $\gamma$  can be found from

$$e^{2\gamma} = \frac{P}{Q}, \quad Q = \Delta + (m^2 + b^2) \sin^2 \theta, \quad P = \Delta + b^2 \sin^2 \theta. \quad (7.15)$$

Let us apply to this solution a magnetic Bonnor-type transformation (7.6). The resulting metric will read

$$ds^2 = \left(\frac{P}{\Sigma}\right)^\nu dt^2 - \left(\frac{P\Sigma}{Q^2}\right)^\nu \frac{Q}{\Delta} (dr^2 + \Delta d\theta^2) - \left(\frac{\Sigma}{P}\right)^\nu \Delta \sin^2 \theta d\varphi^2, \quad (7.16)$$

where

$$\nu = \frac{2}{1 + \alpha^2}. \quad (7.17)$$

The corresponding magnetic potential and the dilaton function will be

$$a = \frac{2\sqrt{\nu}mb \cos \theta}{\Sigma}, \quad \phi = -\frac{\alpha\nu}{2} \ln \frac{P}{\Sigma}. \quad (7.18)$$

For  $\alpha = 0$  new solution reduces to the original Bonnor solution for electrovacuum [22].

Similarly, an electric transformation (7.8) applied to (7.12) gives the same metric (7.16) and

$$v = \frac{2\sqrt{\nu}mb \cos \theta}{\Sigma}, \quad \phi = \frac{\alpha\nu}{2} \ln \frac{P}{\Sigma}, \quad (7.19)$$

both solutions are related by the discrete duality (7.3). A magnetic (electric) field coincides with that of a magnetic (electric) dipole

$$p = \frac{2mb}{(1 + \alpha^2)^{1/2}}. \quad (7.20)$$

The Schwarzschild mass is given by

$$M = \frac{2m}{1 + \alpha^2}, \quad (7.21)$$

and the dilaton charge is related to mass as

$$D = \pm \alpha M. \quad (7.22)$$

This relation is similar to that of the extremal dilaton black holes [15].

It is worth noting that for KK case  $\alpha = \sqrt{3}$  the target space metric is not rescaled, as it is clear from Eq. (7.7). In this case the function  $Q$  does not enter the transformed solution (7.15). In the stringy case  $\alpha = 1$  the metric (7.16) is given entirely in terms of rational functions.

## 8 Conclusion

By direct integration of Killing equations for the target space corresponding to the stationary Einstein–Maxwell–dilaton theory we have shown that for general dilaton coupling constant  $\alpha$ , with notable exceptions of  $\alpha = 0$  and  $\alpha = \sqrt{3}$ , only a five-dimensional solvable Killing algebra holds. This algebra is isomorphic to the maximal solvable subalgebra of the  $sl(3, R)$ , to which the symmetry algebra is enlarged in the exceptional case  $\alpha = \sqrt{3}$ . For  $\alpha \neq \sqrt{3}, 0$  the isometry group does not contain the essentially non-trivial Ehlers–Harrison-type transformations and does not possess enough symmetries to render the system to be sharing two-dimensional integrability property of the KK and BMD theories.

In two critical cases  $\alpha = \sqrt{3}$  and  $\alpha = 0$  the target space has the structure of cosets  $SL(3, R)/SO(3)$  and  $(SU(2, 1)/S(U(2) \times U(1))) \times R$  respectively. Both can be parametrized by five real variables in which the corresponding lagrangians have very similar structure. Using this formulation we have found the non-trivial discrete (Bonnor-type) transformations for an arbitrary dilaton coupling constant. Their application (in combination with continuous transformations) leads to new solutions of dilaton gravity.

It may seem rather dissapointing that in the stringy case  $\alpha = 1$  the system does not possess enough symmetries to make the theory to be two-dimensionally integrable. This symmetry breaking by a non-critical dilaton may be attributed to absence of the continuous duality rotation symmetry. This is a special property of the truncated stringy gravity action (1.1) without an axion field. In fact the dilaton has to be considered rather as representing the  $SL(2, R)/U(1)$  coset [23]. Within this larger model the continuous duality rotation *is* a symmetry, and consequently, the isometry group of the corresponding target space is substantially enlarged [16].

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